Simple Linear Time Algorithms For Piercing Pairwise Intersecting Disks[☿]

Ahmad Biniaz[♁] *Prosenjit Bose*[♂] *Yunkai Wang*[♃]

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Abstract. A set $\mathcal{D}$ of disks in the plane is said to be pierced by a point set P if each disk in $\mathcal{D}$ contains a point of P. Any set of pairwise intersecting unit disks can be pierced by 3 points (H. Hadwiger and H. Debrunner, Ausgewählte Einzelprobleme der kombinatorischen Geometrie in der Ebene, Enseignement Math, 1955). Stachó and independently Danzer established that any set of pairwise intersecting arbitrary disks can be pierced by 4 points (L. Stachó, A Gallai-féle körletűzési probléma megoldása, in Matematikai Lapok, 32(1-3), p. 19-47, 1981-84. L. Danzer, Zur Lösung des Gallaischen Problems über Kreisscheiben in der Euklidischen Ebene, Studia Scientiarum Mathematicarum Hungarica, 21, p. 111-134, 1986.) Existing linear-time algorithms for finding a set of 4 or 5 points that pierce pairwise intersecting disks of arbitrary radius use the LP-type problem as a subroutine. We present simple linear-time algorithms for finding 3 points for piercing pairwise intersecting unit disks, and 5 points for piercing pairwise intersecting disks of arbitrary radius. Our algorithms use simple geometric transformations and avoid heavy machinery. We also show that 3 points are sometimes necessary for piercing pairwise intersecting unit disks.

1 Introduction

Let $\mathcal{D}$ be a set of pairwise intersecting disks in the plane. Helly's theorem states that if every set of 3 disks in $\mathcal{D}$ has a non-empty intersection, then all disks in $\mathcal{D}$ can be pierced by 1 point, in other words, $\cap\mathcal{D}$ is non-empty [10, 11]. Finding a piercing point set is more difficult if the disks in $\mathcal{D}$ only intersect pairwise and $\mathcal{D}$ contains groups of 3 disks that have no common intersection. Danzer [4] and Stachó [16] independently showed that such a set $\mathcal{D}$ can be pierced by at most 4 points. Danzer's proof is based on his first unpublished proof in 1956, while Stachó's proof uses similar ideas that were used in his previous construction of 5 piercing points in 1965 [15]. Even though Danzer proved that 4 points are sufficient, the proof is not constructive [4]. Stachó's construction is simpler, but it is still not simple enough to be easily turned into a subquadratic algorithm [15, 16]. Har-Peled et al. [9] presented the first deterministic linear-time algorithm for finding 5 piercing points of a set $\mathcal{D}$ by formulating the piercing problem as an LP-type problem. An LP-type problem is an abstract generalization of a low-dimensional linear program. Chazelle and Matoušek showed that LP-type problems can be solved in deterministic linear time if we have a constant-time violation test and the range space has bounded VC-dimension [3]. More recently, Carmi et al. [2] presented a linear time algorithm for finding 4 piercing points. Their algorithm requires the computation of the smallest

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♁School of Computer Science, University of Windsor, Windsor, Canada. abiniaz@uwindsor.ca
♂School of Computer Science, Carleton University, Ottawa, Canada. jit@scs.carleton.ca
♃Amazon, Vancouver, Canada. yunkaiwang1996@gmail.com

disk that intersects every disk in $\mathcal{D}$, which they formulated as an LP-type problem [3, 13]. They pose as an open problem to find the piercing set without using linear programming.

As for lower bounds on this problem, Grünbaum [6] provides a set of 21 pairwise intersecting disks that cannot be pierced by 3 points. Later, Danzer [4] reduced the number of disks to 10. This is close to optimal since every set of 8 pairwise intersecting disks can be pierced by 3 points [15]. We will give an alternate proof to this result in Section 1.2. Danzer's construction is difficult to verify since the positions of the disks cannot be visualized easily. Har-Peled et al. [9] gave a simpler construction with 13 disks.

Hadwiger and Debrunner [7] showed that if all the disks in $\mathcal{D}$ have the same radius, then 3 points are sufficient to pierce $\mathcal{D}$. Their algorithm computes the smallest regular hexagon enclosing the centers of all disks in $\mathcal{D}$. It is not clear how one can simply find such a hexagon in linear time.

1.1 Our Contributions

We first show that 3 points are always sufficient to pierce 8 pairwise intersecting disks. We then present two deterministic linear time algorithms for finding 3 points that pierce a set of pairwise intersecting *unit disks* (disks of radius one). We also present a set of 9 pairwise intersecting unit disks that cannot be pierced by 2 points. This shows that 3 points are sometimes necessary and always sufficient to pierce pairwise intersecting unit disks. Finally, we present a deterministic linear time algorithm for finding 5 points that pierce a set of pairwise intersecting *arbitrary disks* (disks of arbitrary radii). Most of our algorithms employ elementary geometric transformations and we try to exploit properties of arrangements of pairwise intersecting disks to avoid using LP-type machinery in an effort to keep our algorithms simple.

1.2 Piercing Eight Disks with Three Points

In this section, we will prove that any set of 8 pairwise intersecting arbitrary disks can be pierced by 3 points[1]. Before we come to the proof, we first present a useful geometric observation. We refer to a set of 3 disks that have a common intersection point as a *Helly triple*. If 3 disks do not have a common intersection point, we will refer to this triple as *non-Helly*.

Lemma 1. *Every set of* 4 *disks whose centers are in convex position contains a Helly triple.*

Proof. Let a, b, c and d be the centers of these disks in counterclockwise order along their convex hull. We denote the disk centered at point p as $D(p)$. Let x be a point on line segment ac that lies in the intersection of $D(a)$ and $D(c)$. Let y be a point on line segment bd that lies in the intersection of $D(b)$ and $D(d)$. x splits ac into two line segments, ax and xc, y splits bd into by and yd. Among these four line segments, two of them must cross. With a suitable translation and relabeling, assume ax and by cross. By the Triangle's Inequality, $|ax| + |by| > |ay| + |bx|$. This implies that either ax is larger than ay or by is larger than bx. In the former case, $|ax| > |ay|$ implies that y lies within $D(a)$, so $y \in D(a) \cap D(b) \cap D(d)$. In the latter case, using a similar argument, we can conclude that $x \in D(a) \cap D(b) \cap D(c)$. Figure 1 shows the latter case and $\{D(a), D(b), D(c)\}$ is a Helly triple. □

[1] We note that this result did not appear in the conference version of Har-Peled et al. [8], however, it appeared independently and simultaneously in the masters thesis of the third author [17] and in the journal version of Har-Peled et al. [9].

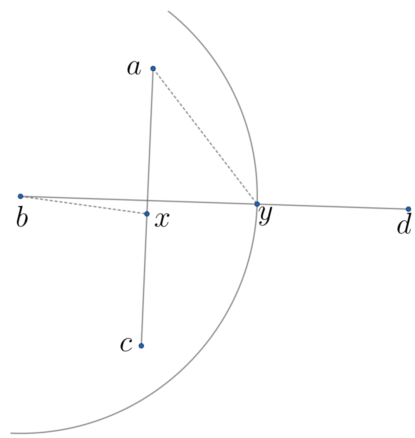


Figure 1: If ax and by intersect and $|by| > |bx|$, then x lies within $D(b)$ and the intersection of $D(a)$, $D(b)$, and $D(c)$ is nonempty.

By the Happy Ending theorem [5], it is known that every set of 5 points contains 4 points that are in convex position. We can then use the above observation to prove the following theorem:

Theorem 1. *Every set of* 8 *pairwise intersecting arbitrary disks can be pierced by at most* 3 *points.*

Proof. By the Happy Ending theorem, we can find 4 points that are in convex position out of the 8 centers. Then by lemma 1, the 4 disks centered at these points contain a Helly triple. Let p_1 be a point that lies in the common intersection of this Helly triple. There now are 5 disks that may not be pierced by p_1. We can again find 4 disks whose centers are in convex position by the Happy Ending theorem. Again, by Lemma 1 we find a new Helly triple among these 4 disks and we let p_2 be a point that lies in the intersection of this new Helly triple. There are now 2 disks that may not be pierced. We choose p_3 to be a point that lies in the common intersection of the two remaining disks. □

2 Piercing Pairwise Intersecting Unit Disks

In this section, we first present our deterministic linear-time algorithms for piercing pairwise intersecting unit disks by 3 points. Let $\mathcal{D}$ be a set of pairwise intersecting unit disks, each disk D_i is centered at $c_i = (x_i, y_i)$. We present two algorithms. The first algorithm finds 3 points that pierces the set $\mathcal{D}$, where the placement of the 3 points is based on the position of the smallest disk that intersects all the disks. However, computing these 3 points in linear-time requires the machinery of efficiently solving LP-type problems. We then give a much simpler linear-time algorithm that finds the 3 points that pierce $\mathcal{D}$ that takes further advantage of the fact that the disks have unit radius. Then we show a set of 9 pairwise intersecting unit disks that cannot be pierced by 2 points. We denote the Euclidean distance between points a and b by $|ab|$.

A point a is to the left (resp. right) of a non-horizontal line l if the intersection point of the horizontal line through a with l lies to the left (resp. right) of a. Similarly, a point a is above (resp. below) a non-vertical line l provided that the intersection point of a vertical line through a with l is above (resp. below) a.

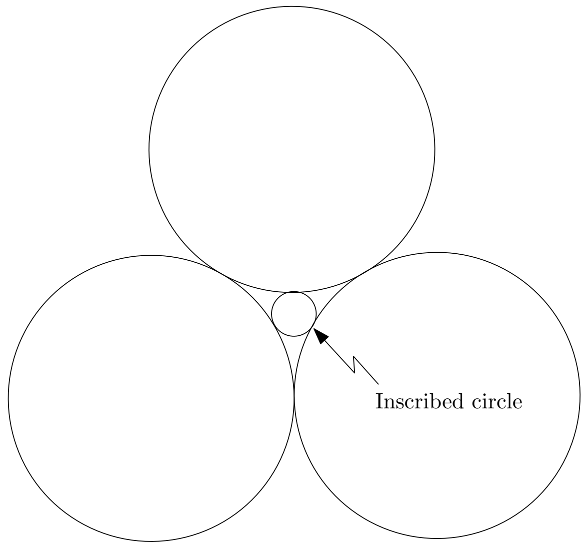


Figure 2: Inscribed circle of a non-Helly triple.

2.1 Algorithm using LP-type Machinery

Let $\{D_1, D_2, D_3\}$ be three unit disks that are pairwise intersecting but are non-Helly. Let D be the smallest disk that is tangent to all three disks. We slightly abuse terminology and refer to D as the *inscribed* circle of the non-Helly triple (see Figure 2). We begin this subsection with the following geometric observation:

Lemma 2. *The inscribed circle of a non-Helly triple of three pairwise intersecting unit disks has radius at most* $\frac{2}{\sqrt{3}} - 1$.

Proof. Let $\{D_1, D_2, D_3\}$ be the three unit disks that are non-Helly and let D be the inscribed circle of these three disks. Let c be the center of D and let r be its radius. $\angle c_1cc_2 + \angle c_2cc_3 + \angle c_1cc_3 = 2\pi$, so there must exist an angle that is at least $2\pi/3$. Without loss of generality, assume $\angle c_1cc_2 \geq 2\pi/3$. $|c_1c| = |c_2c| = 1 + r$, so $|c_1c_2| \geq \sqrt{3}(1 + r)$. Since D_1 and D_2 are two intersecting unit disks, we have $|c_1c_2| \leq 2$. Therefore, $\sqrt{3}(1 + r) \leq 2$ which implies $r \leq \frac{2}{\sqrt{3}} - 1$. □

Löffler and van Kreveld [12] showed that given a set of disks, one can compute the smallest disk that intersects every disk in the set in linear time since this problem is LP-type [3, 13]. We summarize their result below.

Lemma 3. *(Theorem 6 in [12]) Given a set $\mathcal{D}$ of n pairwise intersecting disks in the plane, we can compute the smallest disk that intersects every disk in $\mathcal{D}$ in deterministic $O(n)$ time. Note that if $\mathcal{D}$ is Helly, this smallest disk has zero radius.*

We can use Lemma 2 to prove the following theorem.

Theorem 2. *Let $\mathcal{D}$ be a set of pairwise intersecting unit disks. In $O(|\mathcal{D}|)$ time, we can compute 3 points that pierce $\mathcal{D}$.*

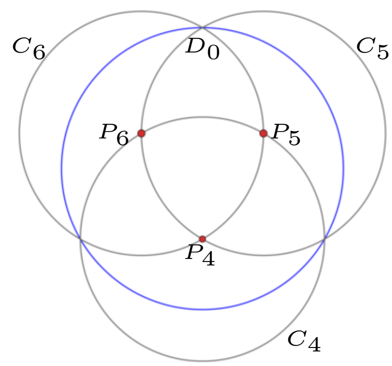


Figure 3: The location of P_4, P_5 and P_6 and how D_0 is covered by C_4, C_5 and C_6.

Proof. We first compute the smallest disk that intersects every disk in $\mathcal{D}$ in deterministic linear time using the LP-type approach outlined in Löffler and van Kreveld [12]. If the radius of this disk is zero, then the piercing point is returned by their algorithm.

Otherwise, suppose that $\mathcal{D}$ is non-Helly and the radius of this disk is greater than zero. Let D be this smallest disk that intersects every disk in $\mathcal{D}$, and let c and r be its center and radius, respectively. Our choice of D ensures that it is tangent to three disks in $\mathcal{D}$; otherwise, the radius of D can be reduced which contradicts minimality. The three disks tangent to D pairwise intersect but are non-Helly. Therefore, D is the inscribed circle of these three disks. By Lemma 2, $r \leq \frac{2}{\sqrt{3}} - 1$. Every disk $D_i \in \mathcal{D}$ with center c_i intersects D, so $|c_i c| \leq \frac{2}{\sqrt{3}}$. Let D_0 be a disk centered at c with radius $\frac{2}{\sqrt{3}}$. By translation, we make c the origin. The centers of all the disks in $\mathcal{D}$ falls in D_0. Let $P_4 = (0, -\frac{1}{\sqrt{3}}), P_5 = (\frac{1}{2}, \frac{1}{2\sqrt{3}}), P_6 = (-\frac{1}{2}, \frac{1}{2\sqrt{3}})$. Let C_4, C_5, and C_6 be three disks of radius 1 and centers P_4, P_5 and P_6, respectively; see Figure 3. The points P_4, P_5, and P_6 pierce every disk in $\mathcal{D}$ since $D_0 \subset C_4 \cup C_5 \cup C_6$. We now show this.

Let A (resp. B) be the intersection point between C_4 and D_0 that falls in the third (resp. fourth) quadrant. Any unit disk whose center falls in $C_4 \cap D_0$ is pierced by P_4. This region is illustrated in red in Figure 4. Now we want to prove that P_5 and P_6 pierce $D_0 \setminus C_4$. Let C_5's intersection point with D_0 other than B be C, so the area illustrated in green in Figure 4 is covered by P_5. The red bounded area and the green bounded area intersects at two points, one is B, and the other one is P_6. P_6 has distance 1 to both point A and point C. So the area $C_0 \setminus \{C_4 \cup C_5\}$ is covered by P_6. $\square$

2.2 Algorithm For Computing Three Piercing Points

The linear time algorithm outlined in the previous subsection found 3 piercing points but used LP-type machinery in order to compute the 3 points. We now present an alternative proof of Theorem 2 which results in a much simpler algorithm. We achieve this by further leveraging the geometry of the situation. We begin with a simple geometric observation that follows from elementary trigonometry.

Observation 1. *Let D be a unit disk centered at the origin. For any θ in the range $(0, \pi/2)$, D can cover a rectangle with height $2\sin(\theta)$ and width $2\cos(\theta)$ when the center of D coincides with the*

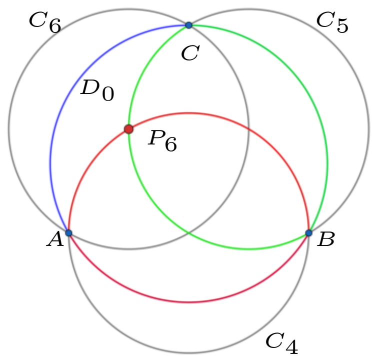


Figure 4: P_4, P_5 and P_6 cover D_0.

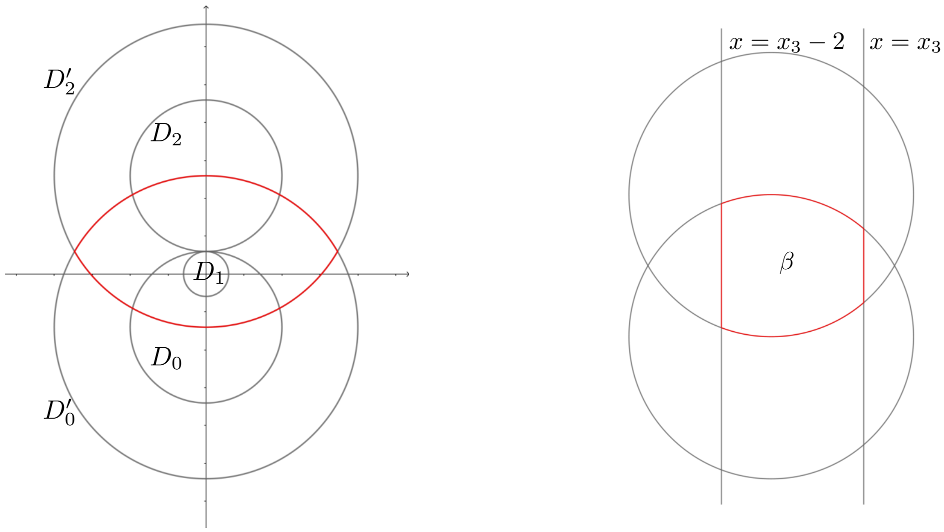
center of the rectangle.

We now present an alternative, simpler linear-time algorithm that can compute 3 points that pierce $\mathcal{D}$.

Theorem 2. *Let $\mathcal{D}$ be a set of pairwise intersecting unit disks. In $O(|\mathcal{D}|)$ time, we can compute 3 points that pierce $\mathcal{D}$.*

Proof. (Alternative proof) Let D_1 be an arbitrary disk in $\mathcal{D}$. We reduce its radius while keeping its center c_1 fixed until D_1 is tangent to another disk $D_2 \in \mathcal{D}$. This can be completed in $O(|\mathcal{D}|)$ time by computing the distance from c_1 to all other disks in $\mathcal{D}$. Notice that after the transformation of D_1, the disks in $\mathcal{D}$ are still pairwise intersecting and any set of points that pierces the new set of disks also pierces the original set of disks. Let r_1 be the radius of D_1. After this transformation, $r_1 \leq 1$, and D_1 is tangent to D_2. By a translation and rotation of the set $\mathcal{D}$, we move c_1 to the origin and c_2 to a point that lies on the positive y-axis with coordinate $(0, r_1 + 1)$. Let D_0 be a unit disk (not necessarily in $\mathcal{D}$) with center $c_0 = (0, r_1 - 1)$. Since $r_1 \leq 1$, $D_1 \subseteq D_0$. Any disk that intersects D_1 also intersects D_0. Let D_0' and D_2' be two disks with radius 2 and centers c_0 and c_2, respectively. We will refer to $D_0' \cap D_2'$ as the *lens* formed by these two disks. See Figure 5(a) where the boundary of the lens is highlighted in red. If a unit disk D_i intersects D_0 and D_2, then $|c_0 c_i| \leq 2$, $|c_2 c_i| \leq 2$ and $c_i \in D_0' \cap D_2'$. Thus, every unit disk in $\mathcal{D}$ has its center in the lens. We say an area is *covered* by a point set P if every point in the area has distance at most 1 to at least 1 point in P. If we cover the lens with a set of points P, then every disk in $\mathcal{D}$ will be pierced. It is not possible to cover the lens with 3 points, however, since the diameter of the lens is $2\sqrt{3}$, the centers lie in a restricted subregion of the lens. We show how to cover this restricted region with 3 points. Let $\beta(a, b)$ represent the set of points in $D_0' \cap D_2'$ whose x-coordinate lies in the interval $[a, b]$ where $a \leq b$ and $a, b \in [-\sqrt{3}, \sqrt{3}]$. When the values of a and b are clear from the context, we will refer to the region $\beta(a, b)$ as β.

Let D_3 be the disk in $\mathcal{D}$ whose center has maximum x-coordinate. In the sequel, we let x_i be the x-coordinate of the center c_i of disk D_i. Since D_3 belongs to $\mathcal{D}$, it must intersect D_1 and D_2. We note that by the maximality of x_3, we have $x_3 \geq 0$ since $x_3 \geq x_1 = 0$. By construction, the boundaries of D_0' and D_2' intersect at two points: $(\sqrt{3}, r_1)$ and $(-\sqrt{3}, r_1)$. Thus, c_3 must either fall



(a) The proof set up. (b) Area that we need to cover.

Figure 5: Illustration of the proof of Theorem 2.

on or to the left of the line $x = \sqrt{3}$. We conclude that $x_3 \le \sqrt{3}$. Therefore, we have $0 \le x_3 \le \sqrt{3}$. The disk D_3 can be found in $O(|\mathcal{D}|)$ time by verifying the x-coordinate of the center of every disk in $\mathcal{D}$. For every disk $D_i \in \mathcal{D}$, $|c_i c_3| \le 2$ since D_i and D_3 intersect. This also implies that $|x_i x_3| \le 2$ since both D_i and D_3 are unit disks. Therefore, in addition to being in $D_0' \cap D_2'$, the x-coordinate of all the centers lie in the interval $[x_3 - 2, x_3]$. This means that all the centers of the disks in $\mathcal{D}$ lie in the region $\beta(x_3 - 2, x_3)$, which is illustrated in red in Fig 5(b).

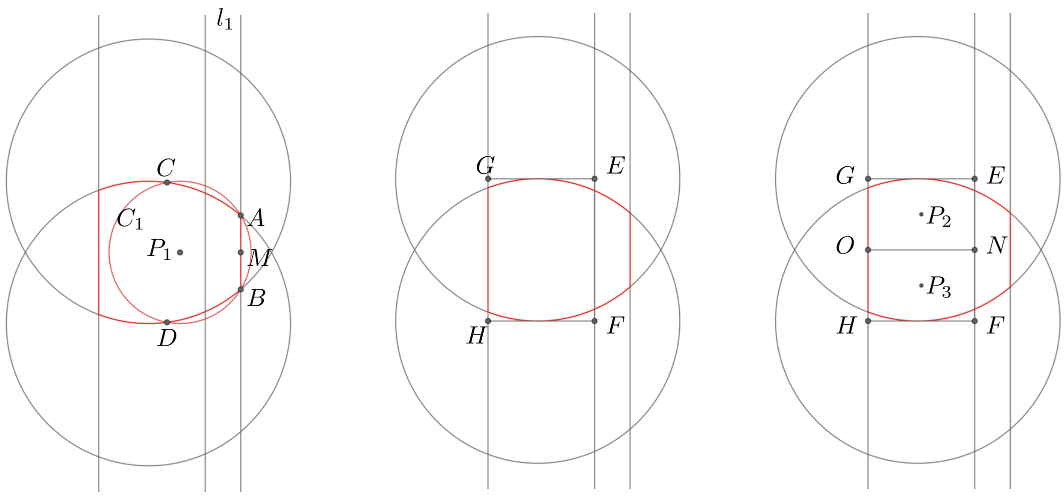
Therefore, if we can find 3 points that cover β, then those three points pierce every disk in $\mathcal{D}$. We now show how to find three points P_1, P_2, P_3 that cover β. As noted above, we have that $0 \le x_3 \le \sqrt{3}$. We consider two cases, namely when $1 \le x_3 \le \sqrt{3}$ and $0 \le x_3 < 1$.

Case 1: $1 \le x_3 \le \sqrt{3}$. To pierce all the disks in $\mathcal{D}$ we need to cover $\beta(x_3 - 2, x_3)$. Let A (resp. B) be the rightmost point of β on the boundary of D_0' (resp. D_2'). The first point P_1 is chosen to be the point that falls in β and has distance 1 to both A and B. Thus, P_1 lies on the bisector of the line segment AB, so by construction, P_1 lies on the line $y = r_1$. Let C_1 be a circle of radius 1 centered at P_1; See Figure 6(a).

Let l_1 be the vertical line $x = x_3 - \frac{1}{2}$. First we prove that P_1 always lies to the left of l_1. Let the midpoint of line segment AB be M. $|AB|$ decreases as x_3 increases; thus, $|AB|$ is maximized when $x_3 = 1$. When $x_3 = 1$, using the equations for D_0' and D_2', we note that $|AB| = 2\sqrt{3} - 2 < \sqrt{3}$. Since $|AB| < \sqrt{3}$, we have that $|AM| < \sqrt{3}/2$. Since $\triangle P_1 AM$ is a right triangle and $|AP_1| = 1$, by the Pythagorean theorem, $|P_1 M| > 1/2$. Therefore, P_1 lies to the left of l_1.

Let the intersection point of circle C_1 and D_0' different from A be labelled C, and the intersection point of circle C_1 and D_2' different from B be labelled D. Recall that the y-coordinate of P_1 is r_1. Since C_1 has unit radius, it is tangent to both lines $y = r_1 + 1$ and $y = r_1 - 1$. By construction, we also have that D_0' is tangent to the line $y = r_1 + 1$ and D_2' is tangent to the line $y = r_1 - 1$. Since the x-coordinate of P_1 is at least zero and the circle C_1 is tangent to these two lines, both C and D lie on or to the left of the vertical line through P_1. If the x-coordinate of P_1 is zero, then both C and D are the points of tangency and thus lie on the vertical line through P_1. Otherwise, they lie to the left of the line. See Figure 6(a).

Since the radius of C_1 is 1, the radius of D_0' is 2, and the point C lies to the left of l_1, we have that the clockwise arc from C to A on the boundary of D_0' and the clockwise arc from B to D on the boundary of D_2' are both contained in C_1. Therefore, the center of any unit disk of $\mathcal{D}$ that lies on or to the right of l_1 is contained in the disk C_1. We now show how to compute points P_2 and P_3 to pierce all the disks that do not contain P_1, namely the disks in $\mathcal{D}$ whose centers are in β but outside disk C_1. The exact coordinates of A, B, P_1, P_2, and P_3 are given in Appendix A.



(a) Location of P_1. (b) Remaining area to be covered. (c) Location of P_2 and P_3.

Figure 6: Illustration of the proof of Theorem 2.

Consider the rectangle formed by the following 4 points: $E = (x_3 - \frac{1}{2}, r_1 + 1), F = (x_3 - \frac{1}{2}, r_1 - 1), G = (x_3 - 2, r_1 + 1), H = (x_3 - 2, r_1 - 1)$. See Figure 6(b). Since D_0' is tangent to the line $y = r_1 + 1$ at $(0, r+1)$, and D_2' is tangent to the line $y = r_1 - 1$ at $(0, r-1)$, the area $\beta \cap \{x < x_3 - \frac{1}{2}\}$ as shown in Fig 6(b) is contained completely within the rectangle $EFHG$. If the points P_2 and P_3 cover this rectangle, then we are done. Let N be the midpoint of line segment EF and let O be the midpoint of line segment GH. See Figure 6(c). We choose P_2 to be the center of the rectangle $ENOG$. Since the height of this rectangle is 1 and its width is 3/2, by Observation 1, P_2 covers this rectangle with $\theta = \pi/3$. Select P_3 to be the center of the rectangle $NFHO$. Again, by Observation 1, P_3 covers this rectangle since it has identical height and width to $ENOG$.

Case 2: $0 \le x_3 < 1$. Recall that the β region we need to cover is the set of points in $D_0' \cap D_2'$ whose x-coordinate lies in the interval $[x_3 - 2, x_3]$. Since the leftmost point on the lens formed by D_0' and D_2' has coordinates $(-\sqrt{3}, r_1)$, we note that the left endpoint of this interval cannot be less than $-\sqrt{3}$. Therefore, the left endpoint lies in the range $[-\sqrt{3}, -1]$. If we reflect all the disks about the y-axis, then the x-coordinates of all the disks lies in the interval $[-x_3, |x_3 - 2|]$. Since $x_3 < 1$, we have that $|x_3 - 2| > 1$. Therefore, after reflection, the right endpoint of the interval for β lies in the range $[1, \sqrt{3}]$. This is exactly the range for Case 1.

□

2.3 A Lower Bound

We now present a set of 9 pairwise intersecting unit disks that cannot be pierced by 2 points. See Figure 7 for an illustration of these disks in a nutshell; details are given in the proof of Theorem 3.

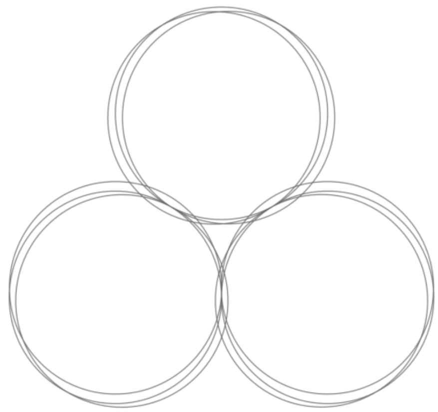


Figure 7: Nine unit disks that cannot be pierced by 2 points.

Theorem 3. *There exists a set of* 9 *pairwise intersecting unit disks that cannot be pierced by* 2 *points.*

Proof. Follow Figure 8. We begin the construction by placing 3 unit disks D_1, D_2, D_3 centered at $(0,0), (2,0), (1,\sqrt{3})$ respectively. These points are the vertices of an equilateral triangle with side length 2. Notice that these disks are pairwise tangent. We denote the center of D_i by c_i. Let C_i be the circle of radius 2 centered at c_i. The intersection of C_1, C_2, and C_3 is a reuleaux triangle, which is illustrated in red in Figure 8. The center of any unit disk, that intersects D_i, lies in C_i. Therefore the center of any unit disk, that intersects the three disks D_1, D_2, and D_3, lies in the reuleaux triangle. We then introduce 6 more unit disks as follows where $\epsilon = 0.01$:

- D_1' with center $c_1' = (2-\sqrt{4-\epsilon^2}, \epsilon)$ on C_2.
- D_1'' with center $c_1'' = (\epsilon, \sqrt{3}-\sqrt{4-(\epsilon-1)^2})$ on C_3.
- D_2' with center $c_2' = (2-\epsilon, \sqrt{3}-\sqrt{4-(\epsilon-1)^2})$ on C_3.
- D_2'' with center $c_2'' = (\sqrt{4-\epsilon^2}, \epsilon)$ on C_1.
- D_3' with center $c_3' = (1+\epsilon, \sqrt{4-(1+\epsilon)^2})$ on C_1.
- D_3'' with center $c_3'' = (1-\epsilon, \sqrt{4-(1+\epsilon)^2})$ on C_2.

We show that $\mathcal{D} = \{D_1, D_1', D_1'', D_2, D_2', D_2'', D_3, D_3', D_3''\}$ is a desired set. Given the above coordinates of the centers of the disks in $\mathcal{D}$, one can verify that by construction, the distance

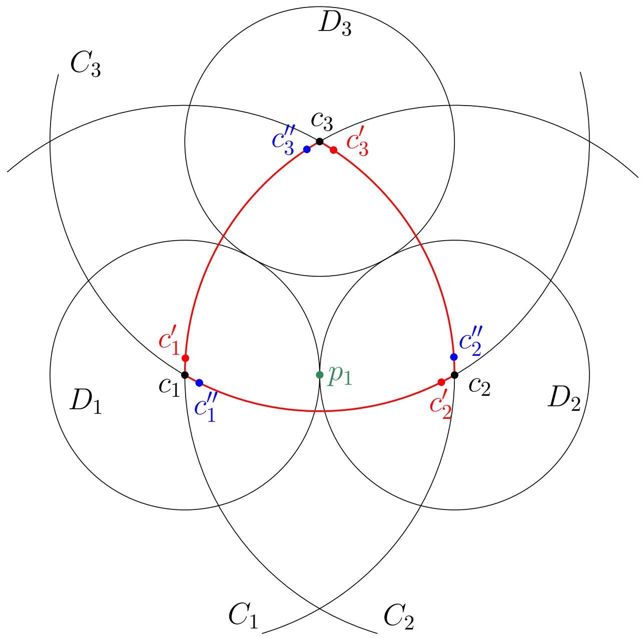


Figure 8: Illustration of the construction of a set of 9 pairwise intersecting unit disks that cannot be pierced by 2 points.

between any two centers is at most 2 and thus the disks are pairwise intersecting. Next, we note that by construction, the disks X, Y, Z with $X \in \{D_1, D_1', D_1''\}$, $Y \in \{D_2, D_2', D_2''\}$, and $Z \in \{D_3, D_3', D_3''\}$ form a non-Helly triple.

Now we show that $\mathcal{D}$ cannot be pierced by two points. For the sake of a contradiction, suppose that points p_1, p_2 pierce all the disks in $\mathcal{D}$. Then one of these points must pierce at least two of the disks D_1, D_2 and D_3 since these three disks form a non-Helly triple. Without loss of generality, we may assume that p_1 pierces D_1 and D_2 (as in Figure 8), and thus $p_1 = (1, 0)$ since $|c_1 c_2| = 2$. By construction, p_1 does not pierce D_1', D_2'', D_3, D_3' and D_3'' since the distance from p_1 to the centers of each of those circles is strictly greater than 1. Thus, these disks must be pierced by p_2, and in particular $p_2 \in D_1' \cap D_2'' \cap D_3$. However, since D_1', D_2'', D_3 is a non-Helly triple, these three disks cannot be pierced by 1 point. □

3 Piercing Pairwise Intersecting Arbitrary Disks

We now consider a set $\mathcal{D}$ of pairwise intersecting disks of arbitrary sizes. Each disk $D_i \in \mathcal{D}$ is described by its center c_i and its radius r_i. Before we introduce our algorithm that computes a set of 5 points that pierce the given set of disks, we first introduce an algorithm that inspired our algorithm. This algorithm computes 7 points that pierce the given set of disks and it does not require solving any LP-type problem.

3.1 Piercing with Seven Points

Even though 4 points are always sufficient to pierce any set of pairwise intersecting disks [4, 15, 16], finding such 4 points is difficult. Carmi et al. [2] presented a linear time algorithm that is quite complex using LP-type machinery. Our goal is to find a simpler algorithm that avoids using heavy machinery but relies more on simple geometric properties. We begin by showing how to find a piercing set of 7 points whose coordinates can be computed in linear time only using simple geometric tranformations. Our proof is a modification of the proof in [1]. The 7 points are the vertices of a regular hexagon and its center.

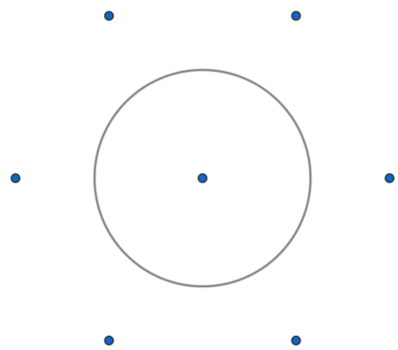


Figure 9: $\{P_1, P_2, P_3, P_4, P_5, P_6, P_7\}$ pierce any disk with radius ≥ 1 and intersects D_1.

Theorem 4. *(Theorem 2 in [1]) Let $\mathcal{D}$ be a set of pairwise intersecting disks in the plane. Then in linear time one can find 7 points that pierce all disks in $\mathcal{D}$.*

Proof. Let $D_1 \in \mathcal{D}$ be the smallest disk in $\mathcal{D}$. Finding this disk takes linear time. By scaling and translation, we assume D_1 is centered at the origin and its radius is 1. Let $P_1 = (0,0)$, $P_2 = (\sqrt{3}, 0)$, $P_3 = (\frac{\sqrt{3}}{2}, \frac{3}{2})$, $P_4 = (-\frac{\sqrt{3}}{2}, \frac{3}{2})$, $P_5 = (-\sqrt{3}, 0)$, $P_6 = (-\frac{\sqrt{3}}{2}, -\frac{3}{2})$, $P_7 = (\frac{\sqrt{3}}{2}, -\frac{3}{2})$ as depicted in Fig 9; the points $P_2, P_3, P_4, P_5, P_6, P_7$ are the vertices of a regular hexagon with sides of length $\sqrt{3}$ centered at the origin. We prove that these 7 points pierce $\mathcal{D}$.

Let $D_i \in \mathcal{D}$ be a disk with center c_i and radius r_i. Since D_1 is the smallest disk in $\mathcal{D}$, we have that $r_i \geq 1$. Since points $P_2, P_3, \ldots, P_7$ are the vertices of a regular hexagon, there must exist a $j \in \{2, 3, \ldots, 7\}$ such that $\angle P_j P_1 c_i \leq \frac{\pi}{6}$. Let $\theta = \angle P_j P_1 c_i$. By the law of cosines,

$$|c_i P_j|^2 = |c_i P_1|^2 + |P_1 P_j|^2 - 2|c_i P_1||P_1 P_j|\cos(\theta)$$

$|P_1 P_j| = \sqrt{3}$ since these points all have distance $\sqrt{3}$ to the origin. $|c_i P_1| \leq r_i + 1$ since D_i and D_1 intersect. We have that $\cos(\theta) \geq \cos(\frac{\pi}{6})$ since $\theta \leq \frac{\pi}{6}$. Therefore, $-2|c_i P_1||P_1 P_j|\cos(\theta) \leq -2|c_i P_1||P_1 P_j|\cos(\frac{\pi}{6})$. By replacing terms in the equation, we get

$$|c_i P_j|^2 \leq |c_i P_1|^2 + (\sqrt{3})^2 - 2\sqrt{3}|c_i P_1|\cos(\frac{\pi}{6})$$

By simplification, we get

$$|c_i P_j|^2 \leq |c_i P_1|^2 + 3 - 3|c_i P_1|$$

By rearranging terms in the equation, we get

$$|c_i P_j|^2 \leq (|c_i P_1| - 1)^2 - |c_i P_1| + 2$$

When $|c_i P_1| \geq 2$, $(|c_i P_1| - 1)^2 - |c_i P_1| + 2 \leq r_i^2 + 2 - |c_i P_1| \leq r_i^2$. Therefore, $|c_i P_j| \leq r_i$ and D_i contains P_j. If $|c_i P_1| \leq 1$, c_i falls in D_1. Then D_i is pierced by P_1 since $r_i \geq 1$. When $1 < |c_i P_1| < 2$ then we have that $(|c_i P_1| - 1)^2 - |c_i P_1| + 2 \leq 1$, therefore $|c_i P_j| \leq 1$ which implies that P_j pierces D_i since $r_i \geq 1$.

□

3.2 Piercing with Five Points

In this section we present a simple linear-time algorithm for piercing pairwise intersecting disks with five points. Recall that each disk $D_i \in \mathcal{D}$ is described by its center c_i and its radius r_i. Let D_1 be the smallest disk in $\mathcal{D}$. We shrink D_1 while fixing its center at c_1 until D_1 becomes tangent to another disk, say D_2. This can be done in linear time by computing the distance of c_1 to all other c_i's to determine the minimum amount required to shrink D_1 until it is tangent to another disk. In this new setting, disks in $\mathcal{D}$ are still pairwise intersecting and any set of points that pierces the new set of disks also pierces the original set of disks. After scaling, rotation and translation, assume that D_1 has radius 1 and is centered at the origin and D_2 is centered on the positive y-axis; these transformations can be performed in linear time.

The rest of our algorithm relies on the following two geometric lemmas whose proofs are constructive. We prove these lemmas in Section 3.2.1 and Section 3.2.2. In the lemmas, we let $P_1 = (0,0)$, $P_2 = (\sqrt{3},0)$, $P_3 = (\frac{\sqrt{3}}{2}, \frac{3}{2})$, $P_4 = (-\frac{\sqrt{3}}{2}, \frac{3}{2})$, $P_5 = (-\sqrt{3},0)$ where P_2, P_3, P_4, P_5 are four vertices of a regular hexagon with sides of length $\sqrt{3}$ centered at the origin P_1, as in Figure 10. These are 5 of the 7 points defined in Theorem 4 that are on or above the line $y = 0$. Let $P = \{P_1, P_2, P_3, P_4, P_5\}$.

Lemma 4. *If the radius of D_1 is 1 and the radius of D_2 is at most $5 + 2\sqrt{6}$, then P pierces $\mathcal{D}$.*

Lemma 5. *If the radius of D_1 is 1 and the radius of D_2 is larger than $5 + 2\sqrt{6}$ and P does not pierce $\mathcal{D}$, then we can find in constant time a different set of 5 points that pierces $\mathcal{D}$.*

These two lemmas are sufficient for proving the existence of 5 piercing points for arbitrary disks. Our piercing algorithm is given below

Algorithm:

1. Find the smallest disk $D_1 \in \mathcal{D}$
2. Reduce the radius of D_1 until D_1 is tangent to a disk in $\mathcal{D}$, say D_2
3. By scaling, rotation and translation of $\mathcal{D}$, let the center of D_1 be the origin and the radius of D_1 be 1. Let D_2 be centered on the y-axis above D_1
4. If $r_2 \leq 5 + 2\sqrt{6}$, then return P as a piercing set for $\mathcal{D}$

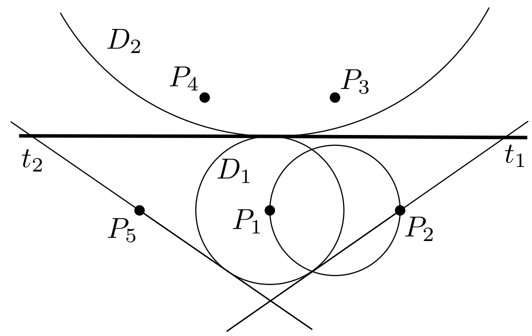


Figure 10: The first candidate set of 5 points and illustration for the proof of Lemma 4.

5. If $r_2 > 5 + 2\sqrt{6}$ and P does not pierce $\mathcal{D}$, then find another set of 5 points by Lemma 5 and return it as a piercing set for $\mathcal{D}$.

Theorem 5. *Given a set of pairwise intersecting arbitrary disks in the plane, in deterministic linear time, we can find 5 points that pierce all the disks.*

Proof. Let $\mathcal{D}$ be a set of pairwise intersecting arbitrary disks. We run the above algorithm on $\mathcal{D}$. If $r_2 \leq 5 + 2\sqrt{6}$, by Lemma 4, P pierces $\mathcal{D}$. If $r_2 > 5 + 2\sqrt{6}$ and there exists at least one disk in $\mathcal{D}$ that is not pierced by any of the 5 points in P, then by Lemma 5 we can find 5 points that pierce $\mathcal{D}$.

The correctness of the algorithm comes from Lemma 4 and Lemma 5, which we prove in Section 3.2.1 and Section 3.2.2, respectively. Step 1 of the algorithm clearly takes linear time. Step 2 can also be completed in linear time by computing the distance from c_1 to all other centers in D. The geometric transformations of Step 3 take linear time. The points P_1 to P_5 can be obtained in constant time after the transformation. Then checking whether these 5 points pierce $\mathcal{D}$ takes linear time. If these 5 points do not pierce $\mathcal{D}$, then by Step 5, we can compute a new set of 5 points that pierce $\mathcal{D}$ in constant time by Lemma 5. □

We now present a few well-known geometric observations about circles that will be used in Section 3.2.1 and Section 3.2.2. These are well-known properties of circles [14].

Observation 2. *Let a, b be two points on an arbitrary line L. For convenience, we rotate and translate L such that it coincides with the x-axis. Without loss of generality, assume that $a < b$. Let D be any disk with radius r whose intersection with the x-axis is the interval $[a, b]$ and whose center c is on or below the x-axis. Denote by D^+ the set of points in D that are on or above the x-axis. All points in D^+ have x-coordinate in the range $[a, b]$ and the highest point in D^+ lies on the line $y = r(1 - \cos(\theta/2))$, where $\theta = \angle acb$. For any other disk C centered on or below the x-axis with radius $r' \geq r$ and whose intersection with the x-axis is $[a', b'] \subseteq [a, b]$, we have that $C^+ \subseteq D^+$. Moreover, if a' and b' are in the open interval (a, b), then the highest point in C^+ is strictly below the line $y = r(1 - \cos(\theta/2))$.*

3.2.1 Proof for Lemma 4

Proof. Recall points P_1 to P_5 where $P_1 = (0, 0)$, $P_2 = (\sqrt{3}, 0)$, $P_3 = (\frac{\sqrt{3}}{2}, \frac{3}{2})$, $P_4 = (-\frac{\sqrt{3}}{2}, \frac{3}{2})$, $P_5 = (-\sqrt{3}, 0)$; see Figure 10. Let t_1 be the line with a positive slope that is tangent to D_1 and passing through P_2. Let t_2 be the line with a negative slope that is tangent to D_1 and passing through P_5. See Figure 10. The equation of t_1 is $t_1 = \frac{\sqrt{2}}{2}x - \frac{\sqrt{6}}{2}$ and the equation of t_2 is $t_2 = -\frac{\sqrt{2}}{2}x - \frac{\sqrt{6}}{2}$. Since D_2 is centered on the positive y-axis, D_2 is tangent to both t_1 and t_2 when $r_2 = 5 + 2\sqrt{6}$. Therefore, when $r_2 \leq 5 + 2\sqrt{6}$, D_2 falls on or above t_1 and t_2.

We now prove that $P_1, \ldots P_5$ pierce $\mathcal{D}$ when $r_2 \leq 5 + 2\sqrt{6}$. It is implied from the proof of Theorem 4 that any disk whose center lies in the first or the second quadrant is pierced by these 5 points. We show that any disk in $\mathcal{D}$ whose center falls in the third or fourth quadrant is pierced by at least one of $\{P_1, P_2, P_5\}$. If all such disks are pierced by at least one of these points, then we are done. So we assume that there exists at least one disk, say D_3, that is not pierced by any of these three points. Without loss of generality, assume that the center c_3 of D_3 lies in the fourth quadrant. We will prove that P_1 or P_2 must pierce D_3. Recall that by construction and transformation, D_2 is tangent to the line $y = 1$ and lies above this line. We will use the line $y = 1$, t_1 and t_2 to derive a contradiction since any disk whose center is below these lines, must intersect these lines to intersect D_2.

Define the interval $[a, b] := D_3 \cap t_1$. If $P_2 \in [a, b]$, then we have a contradiction since $P_2 \in D_3$. Therefore, either $[a, b]$ is strictly to the right or strictly to the left of P_2. We first consider the former case, where $[a, b]$ is strictly to the right of P_2. By construction, D_1 is on or above t_1 and strictly to the left of P_2. By Observation 2, the portion of D_3 on or above t_1 is strictly to the right of P_2. Therefore, D_3 does not intersection D_1 which is a contradiction.

We now consider the case where $[a, b]$ is strictly to the left of P_2. We further refine this case by the location of c_3. Either c_3 is on or to the right of the vertical line through P_2 or it is strictly to the left of this vertical line. In both cases, the contradiction we derive is that D_2 does not intersect D_3.

We start with the case where c_3 is strictly to the left of P_2. In this case, by Observation 2, the highest point that D_3 can reach above the x-axis is on the line $y = \sqrt{3}/2$. However, since $\sqrt{3}/2 < 1$, D_3 does not reach the line $y = 1$ and thus cannot intersect D_2, which is a contradiction.

Finally, we consider the case where c_3 is strictly to the right of P_2. Since $[a, b]$ lies strictly to the left of P_2 on t_1, by Observation 2, the portion of D_3 to the right of P_2 lies below t_1, and thus does not intersect D_2. In addition, the perpendicular projection of c_3 onto the x-axis lies strictly to the right of P_2. This means, by Observation 2, that the portion of D_3 to the left of P_2 lies below the x-axis and thus cannot intersect D_2. Therefore, D_3 does not intersect D_2 which is a contradiction. We conclude that either P_1 or P_2 pierces D_3.

□

3.2.2 Proof for Lemma 5

Proof. Recall the lines t_1, t_2, and the point set P from the proof of Lemma 4. Since $r_2 > 5 + 2\sqrt{6}$, D_2 intersects both t_1 and t_2. Since P does not pierce $\mathcal{D}$, there exists a disk, say $D_3 \in \mathcal{D}$, that is not pierced by P. The disk D_3 intersects both D_1 and D_2. The center c_3 of D_3 cannot lie in the first or second quadrant since otherwise it must contain one point of P as is implied from the proof of Theorem 4. Without loss of generality, we may assume that the center c_3 lies in the fourth quadrant. This is because if all the disks of $\mathcal{D}$ that are not pierced by P only have a center in the third quadrant, then take every disk in $\mathcal{D}$ and reflect its center about the line $x = 0$. Then in this new set, there is a disk that is not pierced by P whose center lies in the fourth quadrant, and the arguments below apply.

The proof proceeds as follows. Since P does not pierce $\mathcal{D}$, we construct in constant time a set P' of 5 points that does pierce $\mathcal{D}$. The locations of the points of P' are based on properties derived from the positions of D_1, D_2 and D_3. We first derive the properties deduced from the fact that D_3 is not pierced by P that are essential in the construction of P'. We then prove that P' pierces $\mathcal{D}$ when P does not.

By Observation 2, since D_3 is not pierced by P_1, D_3 can only intersect D_2 on the right side of the y-axis. This setting is depicted in Figure 11(a). Since the interior of D_1 lies completely below the line $y = 1$ and the interior of D_2 lies completely above this line, any disk in $\mathcal{D} \setminus \{D_1, D_2\}$ must intersect this line in order to intersect both D_1 and D_2.

Define the polygonal line

$$\ell : \begin{cases} y = 0, & x \le \sqrt{3} \\ t_1, & x > \sqrt{3} \end{cases}$$

as shown in Figure 11(a). We prove the following claim about ℓ.

Claim 1. *If D_3 is a disk in $\mathcal{D}$ that is not pierced by P and whose center lies in the fourth quadrant then D_3 lies strictly below ℓ.*

Proof. Recall that we assume, without loss of generality, that c_3 is in the fourth quadrant. Let a, b be the leftmost and rightmost point of $D_3 \cap \ell$, respectively. If a is on the line $y = 0$ and b is on t_1, then by convexity of D_3, P_2 is contained in D_3 which is a contradiction. Therefore, we only need to consider the case where a, b are both on the line $y = 0$ or they are both on the line t_1.

In the former case, by Observation 2, the highest point that D_3 can reach above the x-axis is on the line $y = \sqrt{3}/2$. However, since $\sqrt{3}/2 < 1$, D_3 does not reach the line $y = 1$ and thus cannot intersect D_2, which is a contradiction.

In the latter case, by construction, D_1 is on or above t_1 and strictly to the left of P_2. By Observation 2 the portion of D_3 on or above t_1 is strictly to the right of P_2. Therefore, D_3 does not intersection D_1 which is a contradiction.

Therefore, we conclude that D_3 does not intersect ℓ and lies strictly below ℓ. □

Claim 1 implies that any disk in $\mathcal{D}$ whose center falls above ℓ must cross ℓ in order to intersect with D_3.

We now construct a point set $P' = \{P_6, P_7, P_8, P_9, P_{10}\}$ and then show that P' pierces $\mathcal{D}$. The location of these points fundamentally relies on the fact that the set $P = \{P_1, P_2, \ldots, P_5\}$ does not pierce $\mathcal{D}$ and repeatedly uses the fact that D_1 and D_2 must each intersect D_3. It also uses the fact that D_3 is not pierced by P and that D_3 lies below ℓ. We begin by setting $P_6 = (0, -3)$. We now show how to construct P_7, P_8, P_9, and P_{10}. The method of construction gives certain geometric properties to these points which are then used to show why they pierce $\mathcal{D}$ when P does not. The approximate coordinates of these points appears in the table below and the derivation of the coordinates of these points is given in Appendix B.

- **Computing P_7:** Let C_1 (resp. C_2) be the circle passing through P_6 that is tangent to disk D_1 and line $y = 1$ in the left side (resp. right side) of the y-axis, as in Figure 11(b). Let C_3 be the circle that is centered above $y = 1$ and that is tangent to the disk D_1, the line t_1 and to the x-axis. The disks C_1 and C_3 intersect at two points, where we pick the intersection point that is closer to the origin as the point P_7; see Figure 11(c).
- **Computing P_8:** Now let C_4 be a circle of radius 1 that passes though P_7 and that is tangent to the x-axis, and let C_5 be a circle of radius 1 that passes through P_7 and that is tangent to the line $y = 1$. The point P_8 is the intersection point between C_4 and C_5 that is different from P_7. See Figure 11(d) for an illustration.
- **Computing P_9:** Let C_6 be a circle of radius 1 that passes through P_8 and that is tangent to the line $y = 1$. The intersection point of C_2 and C_6 that falls in the first quadrant is P_9, as depicted in Figure 11(e).
- **Computing P_{10}:** Consider a circle C_7 of radius 1 through P_9 and tangent to D_1. The point P_{10} is the intersection point of C_3 and C_7 that is closer to the origin, as in Figure 11(f).

Points	P_6	P_7	P_8	P_9	P_{10}
x-coord	0	−2.139	−0.410	1.473	0.696
y-coord	−3	0.541	0.459	0.101	1.231

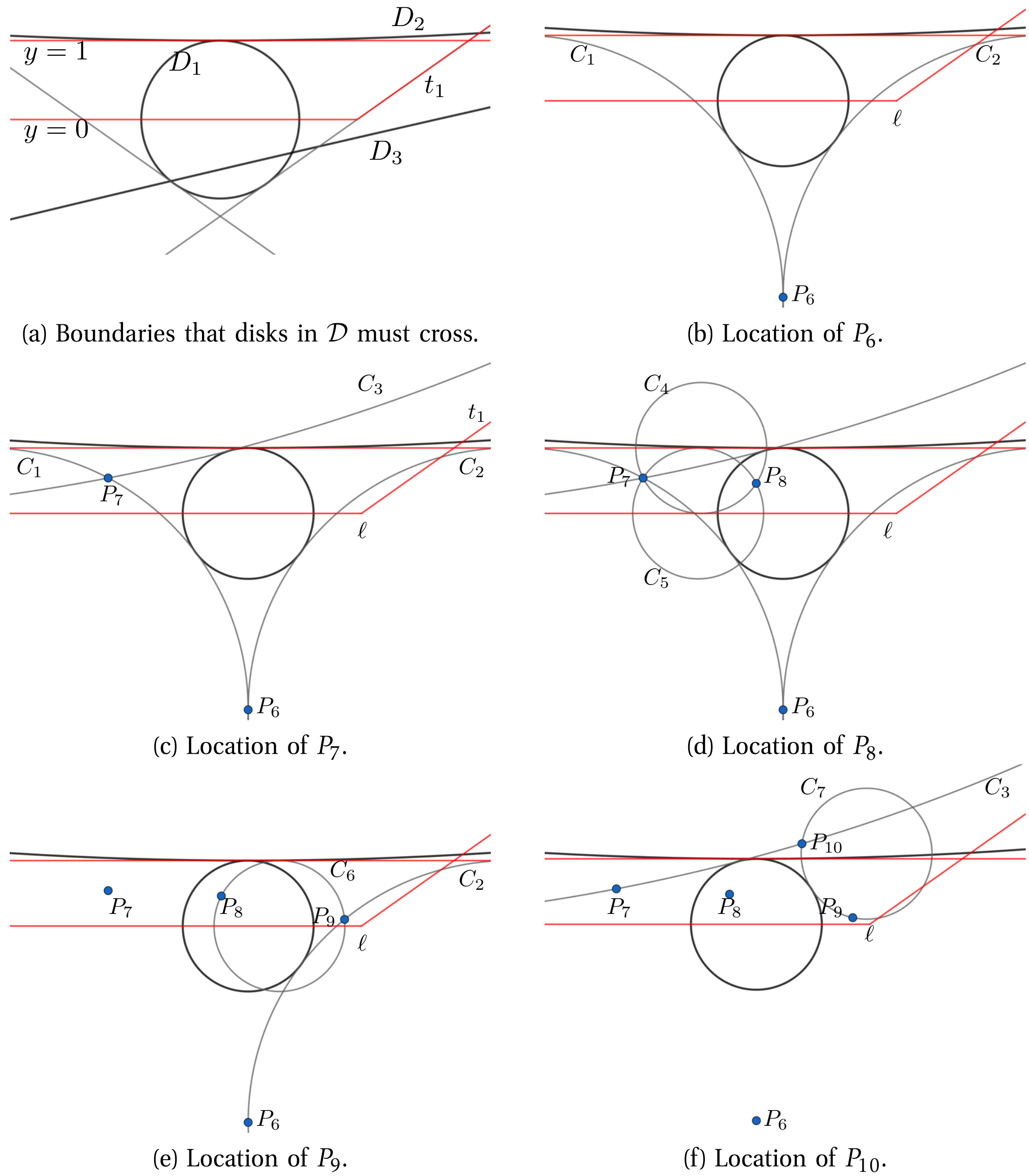
Table 1: Approximate coordinates of the piercing points

Now that all five points in P' have been introduced, we prove that these five points pierce all disks in $\mathcal{D}$. Consider the convex quadrilateral Q formed by P_6, P_7, P_9, and P_{10}, as in Figure 12. We begin by showing that these four points pierce any disk of $\mathcal{D}$ whose center lies outside the quadrilateral.

By construction, since the disk D_1 is contained inside Q, any disk D whose center is outside Q must intersect the boundary of Q in order to intersect D_1. Suppose that there is a disk $D_4 \in \mathcal{D}$ whose center c_4 is outside Q and that is not pierced by P'. We note that D_4 can only intersect one edge of Q, otherwise, by the fact that c_4 is outside Q and convexity of disks and Q, D_4 will contain one of the vertices of Q in its interior which is a contradiction. The key behind this proof is that the geometric properties of the circles $C_1 \ldots C_7$ in addition to the property of ℓ shown in Claim 1 allows us to prove that P' pierces $\mathcal{D}$. We now consider the four cases where D_4 strictly intersects the edge P_7P_{10}, P_6P_7, P_6P_9 or P_9P_{10}, respectively. In each case, we will show that D_4 must contain one of the four vertices of Q in its interior, otherwise, D_4 violates the fact that it intersects every disk in $\mathcal{D}$.

Case D_4 properly intersects P_7P_{10}: Since c_4 is outside Q and D_4 properly intersects P_7P_{10}, c_4 must lie above the line L through P_7P_{10}. By construction, C_3 is tangent to ℓ and D_1. Therefore, since D_4 intersects D_1, we note that D_4 must intersect the boundary of C_3 in order to intersect D_1. If D_4 is tangent to C_3, then it is completely contained in C_3 and thus does not intersect ℓ, which is a contradiction.

Now, consider the path P_7, P_8, P_{10}. Since D_4 properly intersects P_7P_{10}, in order to intersect ℓ, it must intersect P_7P_8 or intersect P_8P_{10} or intersect both. We address each of these subcases in turn. We first address the subcase where D_4 intersects both P_7P_8 and P_8P_{10}. Let C be the unique circle through P_7, P_8, P_{10}. By construction C does not intersect ℓ. Therefore, since D_4 intersects both P_7P_8 and P_8P_{10}, the portion of D_4 below the line through P_7P_{10} is contained in C by Observation 2. Therefore, D_4 does not intersect ℓ in this case, which is a contradiction.



(a) Boundaries that disks in $\mathcal{D}$ must cross.

(b) Location of P_6.

(c) Location of P_7.

(d) Location of P_8.

(e) Location of P_9.

(f) Location of P_{10}.

Figure 11: Illustration of the proof for Lemma 5.

We next address the subcase where D_4 properly intersects P_7P_8. By construction, C_4 is a unit disk whose center lies on the line $y = 1$ with P_7 and P_8 on its boundary. Let C_4^- be the portion of C_4 that lies on or below the line through P_7P_8. Let D_4^- be defined anagolously. By Observation 2, we have that $D_4^- \subset C_4^-$, which means that D_4 does not intersect ℓ, which is a contradiction since D_4 intersects D_3. A similar argument shows that for the subcase where D_4 properly intersect P_8P_{10}, we obtain a contradiction since the unit disk with P_8 and P_{10} on its boundary with center lying above the line through P_8P_{10} does not intersect ℓ.

Case D_4 **properly intersects** P_6P_7: Both P_6 and P_7 lie on C_1, and C_1 is tangent to the line $y = 1$. This means that if D_4 intersects the line $y = 1$, it must properly intersect the segment P_7P_8. By construction, we have that C_5 is a unit disk with P_7 and P_8 on its boundary that is tangent to and lies below the line $y = 1$. Let C_5^+ be the portion of C_5 that lies on or above the line through P_7P_8. Let D_4^+ be defined anagolously. By Observation 2, we have that $D_4^+ \subset C_5^+$, which means that D_4 does not intersect the line $y = 1$, which is a contradiction since D_4 intersects D_2.

Case D_4 **properly intersects** P_6P_9: Both P_6 and P_9 lie on C_2, and C_2 is tangent to the line $y = 1$. This means that if D_4 intersects the line $y = 1$, it must properly intersect the segment P_8P_9. By construction, we have that C_6 is a unit disk that is tangent to and lies below the line $y = 1$ and has P_8 and P_9 on its boundary. An argument similar to the one in the previous case allows us to conclude, using Observation 2, that D_4 cannot intersect the line $y = 1$. This is a contradiction since D_4 intersects D_2.

Case D_4 **properly intersects** P_9P_{10}: Any disk that intersects D_1 between P_9 and P_{10} must contain one of these two points. Otherwise, by Observation 2, its radius is smaller than 1, since C_7 is a unit disk with P_9 and P_{10} on its boundary tangent to D_1. We note that D_4 has radius at least 1 since D_1 has radius 1 and is the smallest disk in $\mathcal{D}$. By Observation 2, we conclude that D_4 must contain P_9 or P_{10} since it intersects D_1.

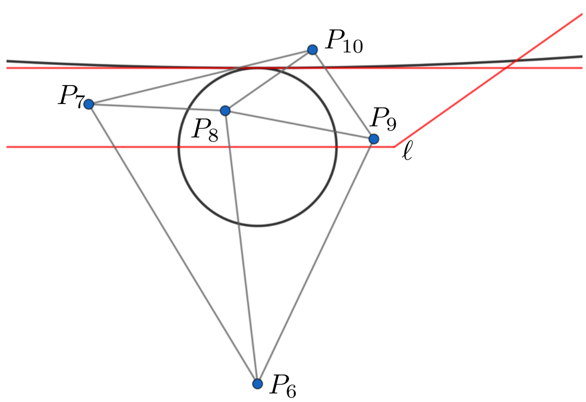


Figure 12: The points P_6, P_7, P_9, P_{10} form a quadrilateral that contains D_1.

Now we show how the disks of $\mathcal{D}$ centered inside the quadrilateral are pierced by points in P'. We divide the quadrilateral into four triangles, as in Figure 12 and look at the case when the center of D_4 lies in each of these triangles.

Case $c_4 \in \triangle P_6P_7P_8$: In this case, in order for D_4 to intersect the line $y = 1$, it must properly intersect P_7P_8. By construction, C_5 is a unit disk whose center lies on the line $y = 0$ with P_7 and

P_8 on its boundary. By Observation 2, we have that any disk whose radius is at least 1 and properly intersects P_7P_8 cannot intersect the line $y = 1$. Moreover, it is not possible for D_4 to intersect $y = 1$ to the left of P_7 because otherwise its center would not be in Q. Therefore, we have a contradiction.

Case $c_4 \in \triangle P_6P_8P_9$: An argument similar to the previous case applies here with the circle C_6 playing the role played by circle C_5 in the previous case.

Case $c_4 \in \triangle P_7P_8P_{10}$: To intersect ℓ in this case, D_4 must intersect the segment P_7P_8 or the segment P_8P_{10}. The arguments presented in the case where D_4 properly intersects P_7P_{10} apply here since D_4 has radius at least 1.

Case $c_4 \in \triangle P_8P_9P_{10}$: Any disk whose center lies in $\triangle P_8P_9P_{10}$ must contain one of these three vertices because the diameter of this triangle is at most 2.

We conclude that any disk D_4 is pierced by at least one point in P'. Also D_1, D_2 and D_3 are pierced by P_8, P_{10}, and P_6, respectively.

Given D_1, D_2, t_1, and t_2, the point set P' can be found in constant time. □

4 Conclusion

In this paper, we gave two simple linear time algorithms for finding 3 piercing points and 5 piercing points for pairwise intersecting unit disks and pairwise intersecting arbitrary disks, respectively. However, it is still not known whether we can find an algorithm for finding a piercing point set of size 4 for any set of pairwise intersecting arbitrary disks without solving an LP-type problem. For the lower bound, the remaining open question is whether any set of 9 pairwise intersecting disks can be pierced by 3 points or not, as it is known that any set of 8 pairwise intersecting disks can be pierced by 3 points [15]. Another interesting open question is whether we can find an efficient algorithm that decides the optimal number of piercing points for any set of pairwise intersecting arbitrary disks.

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A Coordinates of points in Theorem 2

Here are the coordinates of points in the proof of Theorem 2:

$$A = \left(x_3, \sqrt{4 - x_3^2} + r_1 - 1\right)$$

$$B = \left(x_3, -\sqrt{4 - x_3^2} + r_1 + 1\right)$$

$$P_1 = \left(x_3 - \sqrt{2\sqrt{4 - x_3^2} + x_3^2 - 4}, r_1\right)$$

$$P_2 = \left(x_3 - \frac{5}{4}, r_1 + \frac{1}{2}\right)$$

$$P_3 = \left(x_3 - \frac{5}{4}, r_1 - \frac{1}{2}\right)$$

B Coordinates of points in Lemma 5

For each point P_i, let x_i be its x-coordinate and y_i be its y-coordinate, and for each circle C_i, let (x_i', y_i') be its center and r_i' be its radius. We summarize these coordinates in the tables below, followed by the derivations of the coordinates of points P_i and equations of circles C_i[2]:

Points	P_6	P_7	P_8	P_9	P_{10}
x-coord	0	−2.139	−0.410	1.473	0.696
y-coord	−3	0.541	0.459	0.101	1.231

Table 2: Approximate coordinates of the piercing points

Circles	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x-coord of center	−4	4	−7.815	−1.251	−1.298	0.478	1.684
y-coord of center	−3	−3	30.038	1	0	0	1.078
radius	4	4	30.038	1	1	1	1

Table 3: Approximate coordinates of the centers and radii of the circles used to compute P_i's.

$P_6 = (0, -3)$

$C_1 : (x + 4)^2 + (y + 3)^2 = 16$, where $x_1' = -4$, $y_1' = -3$ and $r_1' = 4$.

$C_2 : (x - 4)^2 + (y + 3)^2 = 16$, where $x_2' = 4$, $y_1' = -3$ and $r_2' = 4$.

[2]We provide approximate coordinates in addition to the exact coordinates since the exact coordinates, although simple to compute, become quite messy with many radicals. The approximate coordinates are easier to understand and the arguments presented in the proof of Lemma 5 still hold with the approximate coordinates presented here

$C_3 : (x - x'_3)^2 + (y - y'_3)^2 = (r'_3)^2$, where the equations for the coordinates of the center of C_3 are $x'_3 = -\sqrt{1 + 2r'_3}$ since C_3 is tangent to D_1 and $y'_3 = r'_3$ since C_3 is tangent to the x-axis. Given the fact that C_3 is tangent to the x-axis, tangent to D_1 and tangent to the line t_1, there are three possible disks that satisfy these equations. One where the center is in the first quadrant, one where the center is in the second quadrant and one where the center is in the fourth quadrant. For the proof, we want the one where the center is in the second quadrant. Specifically, we have $x'_3 = -\sqrt{17 + 6\sqrt{6} + 4\sqrt{27 + 11\sqrt{6}}} \approx -7.81509$ and $y'_3 = r'_3 = 8 + 3\sqrt{6} + 2\sqrt{27 + 11\sqrt{6}} \approx 30.0378$.

We obtain P_7 by intersecting C_1 with C_3 and taking the intersection closest to D_1. The exact form for the solution is easily obtained from the equations of C_1 and C_3. We first provide an approximate solution: $P_7 \approx (-2.139, 0.541)$.

The exact coordinates for P_7 are:

$$P_7 = \left(\frac{(-2r'_3 - 6)y_7 + (x'_3)^2 - 9}{2x'_3 + 8}, \frac{-b_7 + \sqrt{b_7^2 - 4a_7c_7}}{2a_7} \right)$$

$$a_7 = (-2r'_3 - 6)^2 + (2x'_3 + 8)^2$$

$$b_7 = 2(-2r'_3 - 6)\left((x'_3)^2 - 9\right) + 8(2x'_3 + 8)(-2r'_3 - 6) + 6(2x'_3 + 8)^2$$

$$c_7 = \left((x'_3)^2 - 9\right)^2 + 8(2x'_3 + 8)\left((x'_3)^2 - 9\right) + 9\,(2x'_3 + 8)^2$$

To compute P_8, we consider two unit disks C_4 and C_5 centered on the lines $y = 1$ and $y = 0$, respectively, such that P_7 is on the boundary of both C_4 and C_5. Since P_7 is on C_4, we have that x'_4 is the solution to the equation $(x'_4 - x_7)^2 + (1 - y_7)^2 = 1$. Thus, $(x'_4, y'_4) \approx (-1.251, 1)$. With a similar reasoning, we get that $(x'_5, y'_5) \approx (-1.298, 0)$. Given the coordinates of the centers of the unit disks C_4 and C_5, we compute the other intersection point: $P_8 \approx (-0.410, 0.459)$. The exact coordinates are below:

$$C_4 : \left(x - \sqrt{2y_7 - y_7^2} - x_7\right)^2 + (y - 1)^2 = 1$$

$$C_5 : \left(x - \sqrt{1 - y_7^2} - x_7\right)^2 + y^2 = 1$$

$$P_8 = \left(\frac{2y_8 + q_1}{q_2}, \frac{-b_8 - \sqrt{b_8^2 - 4a_8c_8}}{2a_8} \right)$$

$$q_1 = \left(\sqrt{1 - y_7^2} + x_7\right)^2 - \left(-\sqrt{2y_7 - y_7^2} - x_7\right)^2 - 1$$

$$q_2 = 2\left(\sqrt{1 - y_7^2} + x_7\right) - 2\left(\sqrt{2y_7 - y_7^2} + x_7\right)$$

$$a_8 = 4 + q_2^2$$

$$b_8 = 4q_1 - 4q_2\left(\sqrt{1 - y_7^2} + x_7\right)$$

$$c_8 = q_1^2 + q_2^2\left(\sqrt{1-y_7^2}+x_7\right)^2 - 2q_1q_2\left(\sqrt{1-y_7^2}+x_7\right) - q_2^2$$

To compute P_9, we compute the center of C_6 which is centered on the line $y = 0$. Given that we have the coordinates of P_8, we note that $(x_6', y_6') \approx (0.478, 0)$. To obtain P_9, we compute the intersection of C_6 with C_2 and take the intersection point closest to the line $y = 1$. Thus, $P_9 \approx (1.473, 0.101)$. The exact coordinates are below:

$$C_6 : \left(x - \sqrt{1-y_8^2} - x_8\right)^2 + y^2 = 1$$

$$P_9 = \left(\frac{-b_9 + \sqrt{b_9^2 - 4a_9c_9}}{2a_9}, \frac{q_3x_9 + q_4}{6}\right)$$

$$q_3 = 8 - 2\left(\sqrt{1-y_8^2} + x_8\right)$$

$$q_4 = \left(\sqrt{1-y_8^2} + x_8\right)^2 - 10$$

$$a_9 = 36 + q_3^2$$

$$b_9 = 2q_3q_4 + 36q_3 - 288$$

$$c_9 = q_4^2 + 36q_4 + 324$$

Since C_7 is a unit disk with P_9 on its boundary and tangent to D_1, with the coordinates of P_9, we obtain that $(x_7', y_7') \approx (1.684, 1.078)$. Finally, by intersecting C_7 with C_3, we obtain $P_{10} \approx (0.696, 1.231)$. The exact coordinates are below:

C_7 is centered at

$$\left(\sqrt{4-(y_7')^2}, \frac{-b_{10} + \sqrt{b_{10}^2 - 4a_{10}c_{10}}}{2a_{10}}\right)$$

$$a_{10} = 4x_9^2 + 4y_9^2$$

$$b_{10} = -4y_9(x_9^2 + y_9^2 + 3)$$

$$c_{10} = \left(x_9^2 + y_9^2 + 3\right)^2 - 16x_9^2$$

$$P_{10} = \left(x_7' - \sqrt{1-(y_{10}-y_7')^2}, \frac{-b_{11} - \sqrt{b_{11}^2 - 4a_{11}c_{11}}}{2a_{11}}\right)$$

$$q_5 = (x_7')^2 + (y_7')^2 - (x_3')^2 - (y_3')^2 + (r_3')^2 - 1 - (2x_7' - 2x_3')x_7'$$

$$a_{11} = (2y_3' - 2y_7')^2 + (2x_7' - 2x_3')^2$$

$$b_{11} = 2q_5(2y_3' - 2y_7') - 2y_7'(2x_7' - 2x_3')^2$$

$$c_{11} = q_5^2 + \left((y_7')^2 - 1\right)(2x_7' - 2x_3')^2$$